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STABILIZING AND DESTABILIZING EFFECTS IN NON-CONSERVATIVE SYSTEMS*

N.V. BANICHUK, A.S. BRATUS and A.D. MYSHKIS

Developing earlier results /1-6/, an investigation is presented of the equilibrium state of a linear autonomous non-conservative mechanical system perturbed by arbitrarily small dissipative forces. Perturbations due to dissipative forces are classical as defective or ideal according to whether they do or do not exceed a critical parameter. The structure of the dissipative operators is studied in both cases. Necessary conditions are established for perturbations effected by small forces linear in the system velocities to be ideal or defective. The structure of the matrices determining ideal perturbations is determined, and a formula is derived for the value of the critical stability parameter, called the perturbation defect. Examples are considered.

1. Statement of the problem. Suppose that an unperturbed mechanical system is described by a system of differential equations

$$x^{\cdot\cdot} + A(p)x = 0 \tag{1.1}$$

where $x = x(t)$ is a vector-valued function with $n \geqslant 2$ components, and A is a real matrix which is an analytic function of a real parameter p. Then the squared frequencies and amplitudes of the normal modes $x = ue^{i\omega t}$ are the eigenvalues and eigenvectors in the problem

$$A(p)u = \omega^2 u \tag{1.2}$$

We will make a few additional assumptions.

1°. A real number p_0 exists such that the eigenvalues $\omega_1^2(p)$ and $\omega_2^2(p)$ for $p < p_0$ are positive and simple; for every $p > p_0$ at least one of them is negative or not real; at $p = p_0$ system (1.2) has a double eigenvalue $\omega_1^2(p_0) = \omega_2^2(p_0) = \tau_1^2 > 0$ to which there corresponds a single eigenvector u_1^0.

2°. All other eigenvalues $\omega_3^2(p), \omega_4^2(p), \ldots, \omega_n^2(p)$ of problem (1.2) for $p \leqslant p_0$ are real, simple and positive:

$$\omega_j^2 = \tau_j^2 > 0 \quad (j = 3, 4, \ldots, n).$$

Condition 1° implies that $A(p_0)$ is not a symmetric matrix.

We note that non-symmetric matrices typically possess a multiple eigenvalue corresponding to a single eigenvector - the phenomenon may be observed in real mechanical systems. It follows from conditions 1° and 2° that when the parameter goes through the value $p = p_0$, system (1.1), previously (non-asymptotically) stable, becomes unstable.

Consider the perturbed system

$$x^{\cdot\cdot} + \varepsilon B(p)x^{\cdot} + A(p)x = 0 \tag{1.3}$$

where $B(p)$ is a real matrix whose elements are analytic functions of p, and $\varepsilon > 0$ is a small parameter. The term $\varepsilon B(p)x^{\cdot}$ is necessary to allow for weak damping. The normal modes of system (1.3) satisfy the equation

$$A(p)u + i\varepsilon\omega B(p)u = \omega^2 u \tag{1.4}$$

For every $\varepsilon > 0$ we define the critical value of p, denoted by p_ε, to be the infimum of the set of p values for which at least one eigenvalue of problem (1.4) has negative imaginary part (if the set is empty we put $p_\varepsilon = -\infty$). The term "critical" is used here in the sense that when $p < p_\varepsilon$ system (1.3) is stable (in any case, if all the $\omega_{j,\varepsilon}$ are simple) whereas it is unstable for all sufficiently small values of the difference $p - p_\varepsilon > 0$.

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The set of points $(\varepsilon, p_\varepsilon)$ in the εp plane is a curve separating the stable region of system (1.3) from its unstable region. We shall call this the critical curve. The real number p_0 figuring in conditions 1° and 2° is clearly the critical p_ε for $\varepsilon = 0$.

Let p_d denote the lower limit of the numbers p_ε as $\varepsilon \to +0$. Its existence follows from our assumption that all the functions in question are analytic.

It is well-known /7/ that an equilibrium stable to potential forces becomes asymptotically stable on applying dissipative forces with total dissipation. Non-conservative systems, however, do not possess this property. In such cases weak damping may have a destablizing effect /1-6/. We know that for the systems considered in this paper $p_d \leqslant p_0$.

The number $d_A(B) = p_0 - p_d$ is known as the (destablization) defect of the matrix B relative to the matrix A. If $d_A(B) = 0$ we say that B is ideal relative to A, and the corresponding perturbations of system (1.1) are called ideal; matrices with $d_A(B) > 0$ will be called defective, and the corresponding perturbations will be called destabilizing perturbations. It has been shown that $p_d \leqslant p_0$ /2, 4-6/. The magnitude of the defect characterizes the destabilizing effect of the term $\varepsilon B(p) x^{\cdot}(t)$ on system (1.1) for small $\varepsilon > 0$.

2. *The equation of the critical curve.* The characteristic equation of system (1.4) is (I is the identity matrix)

$$\Delta = \det (A(p) + i\varepsilon\omega B(p) - \omega^2 I) = 0 \tag{2.1}$$

We are interested in those values of ε, p for which this equation has at least one real solution. Since all quantities occurring in Eq.(2.1), except $i\varepsilon\omega$, are real, we can write

$$\Delta = P(\omega^2, \varepsilon\omega, p) + i\varepsilon\omega Q(\omega^2, \varepsilon\omega, p)$$

Here $P(z, y, p)$ $(Q(z, y, p))$ is a polynomial in z and y of total degree n (resp., $n-1$), of even degree in y, whose coefficient are analytic functions of p. If there exist $\varepsilon > 0$ and p for which Eq.(2.1) has at least one real solution ω, then, as $\omega \neq 0$, these values of ε, p make $P(\omega^2, \varepsilon\omega, p) = 0$, $Q(\omega^2, \varepsilon\omega, p) = 0$. Hence we can elliminate ω, considering, say, the resultant of the left-hand sides of the equation /8, p.187/, considered as a polynomial in ω^2. It is important to observe that since P is the real part of a determinant depending on $i\varepsilon$, P is a polynomial in the even powers of ε. An analogous argument holds for Q. Finally, we obtain an equation of the form

$$R(\varepsilon^2, p) = 0 \tag{2.2}$$

where $R(y, p)$ is a polynomial in y whose coefficients are analytic functions of p. The characteristic Eq.(2.1) has a real solution on every curve defined by Eq.(2.2). It is clear from the definition of the critical value p_ε that $R(\varepsilon^2, p_\varepsilon) = 0$.

Let us call a perturbation of system (1.1) with terms $\varepsilon B x^{\cdot}$ as in (1.3) regular if, for some $\varepsilon_0 > 0$, the set of solutions of Eq.(2.2) corresponding to real values of ω is confined to the strip $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and for every solution of the equation $R(0, p_0^*) = 0$ we have $R_p'(0, p_0^*) \neq 0$.

Henceforth we shall assume that all perturbations are regular. On the critical curve we have $R(0, p_d) = 0$, $R_p'(0, p_d) \neq 0$ (by the definition of p_d as the lower limit of p_ε as $\varepsilon \to +0$). By the implicit function theorem, if ε_0 is sufficiently small the critical curve is defined in the above strip by an equation

$$p_\varepsilon = p_d + \varepsilon^2 p_2 + \varepsilon^4 p_4 + \ldots \tag{2.3}$$

where the right-hand side is analytic. The numbers p_d, p_2, p_4 do not depend on ε; in particular, $p_2 = -R_y'(0, p_d)/R_p'(0, p_d)$. Along the critical curve the solutions of Eq.(2.1) satisfy the condition

$$\operatorname{Im} \omega_i(\varepsilon, p) \geqslant 0, \quad i = 1, 2, \ldots, 2n. \tag{2.4}$$

By a standard theorem, the roots of an algebraic equation are continuous functions of its coefficients. Hence the "left" boundary of the stable region of system (1.3) lies entirely within one of the stable intervals of system (1.1). For every critical curve, therefore, the value of p_d lies within or on the boundary of some such interval.

We also note that, by the implicit function theorem and the continuity theorem for roots of algebraic equations, the critical curve depends continuously on the elements of the matrices A and B. In this sense, therefore, it has the stability property.

Eq.(2.2) defines not only the critical curve but the set of all curves on which the solution of (2.2) is real. The set of all such curves (D-curves) forms a "D-partition" of the strip $0 < \varepsilon < \varepsilon_0$ in the εp plane into regions, within each of which the degree of instability of system (1.3) (i.e., the number of values of ω_j such that $\operatorname{Im} \omega_j < 0$) remains constant: if this number is zero, the system is stable. Henceforth, however, we shall consider only the critical curve itself, which divides the plane into stable and unstable regions.

3. Series expansions. Let $p_d = p_0$. It follows from conditions 1°, 2° and the regularity of the perturbations that the squares of the first two frequencies and the corresponding modes can be expanded in powers of $\varepsilon^{1/2}$ on the critical curve emanating from the point p_0 /9/:

$$\omega_{1,\varepsilon}^2 = \tau_1^2 + \varepsilon^{1/2}\mu_{11} + \varepsilon\mu_{12} + \ldots, u_1^\varepsilon = u_1^0 + \varepsilon^{1/2}u_1^1 + \varepsilon u_1^2 + \ldots \tag{3.1}$$

Here u_1^0 is the unique eigenvector of problem (1.2) corresponding at $p = p_0$ to a double eigenvalue $\omega_1^2(p_0) = \omega_2^2(p_0) = \tau_1^2$.

The remaining perturbed squared frequencies and modes can be expanded in series in powers of ε:

$$\omega_{j,\varepsilon}^2 = \tau_j^2 + \varepsilon\mu_{j1} + \varepsilon^2\mu_{j2} + \ldots, \quad u_j^\varepsilon = u_j^0 + \varepsilon u_j^1 + \varepsilon^2 u_j^2 + \ldots; \quad j = 3, 4, \ldots, n \tag{3.2}$$

where u_j^0 are eigenvectors of problem (1.2) corresponding at $p = p_0$ to eigenvectors $\omega_j^2(p_0) = \tau_j^2$ $(j = 3, 4, \ldots, n)$. We may assume without loss of generality that the eigenvectors are normalized so that

$$(u_j^\varepsilon, u_j^0) = 1, \quad j = 1, 2, \ldots, n \tag{3.3}$$

Here and below parentheses denote the scalar product of real vectors.

From (3.1) and (3.2) we derive expansions for the perturbed frequencies:

$$\omega_{j,\varepsilon} = \pm\tau_j \pm \frac{\varepsilon^\alpha}{2\tau_j}\mu_{j1} \pm \frac{\varepsilon^{2\alpha}}{2\tau_j}\left(\mu_{j2} - \frac{\mu_{j1}^2}{4\tau_j}\right) + \ldots$$
$$\alpha = 1/2, j = 1; \alpha = 1, j = 3, 4, \ldots, n; \tau_j = \sqrt{\omega_{j,0}^2} > 0 \tag{3.4}$$

where the upper sign corresponds to one branch of the function and the lower to the other branch. The quantities μ_{jk} depend on the increments to the initial (unperturbed) eigenvalues ω_j^2 and have to be determined.

Let us expand the coefficients of Eq.(1.4) in powers of p, using the representation (2.3) for $p_d = p_0$. Collecting terms in like powers of ε, we obtain equations for the number μ_{11} and the vectors u_1^1, u_1^2:

$$L_0u_1^0 = 0, \qquad L_0u_1^1 = \mu_{11}u_1^0$$
$$L_0u_1^2 = \mu_{11}u_1^1 + \mu_{12}u_1^0 \mp i\tau_1B_0u_1^0 \tag{3.5}$$
$$(L_0 = A(p_0) - \tau_1^2I,\ A_0 = A(p_0),\ B_0 = B(p_0))$$

For simple eigenvalues,

$$L_0u_j^0 = 0, \quad L_0u_j^1 = \mu_{j1}u_1^0 \mp i\tau_jB_0u_j^0; \quad j = 3, 4, \ldots n \tag{3.6}$$

Eqs.(3.5) and (3.6) were derived using the fact that in (2.3) $p_d = p_0$, i.e., on the assumption that the perturbation corresponding to the matrix B is ideal. If $p_d < p_0$, i.e., the perturbation is defective, it follows from our assumptions that the only eigenvalues corresponding to p_d are simple. Hence the expansions of the required quantities in powers of ε will indeed have the form (3.2), with coefficients as determined from (3.6), with p_0 replaced by p_d.

4. Necessary condition for an ideal perturbation.

Proposition 1. If B realizes an ideal perturbation of system (1.1) as system (1.3), then

$$(B_0u_1^0,\ v_1^0) = 0 \tag{4.1}$$

where u_1^0 is an eigenvector of problem (1.2) corresponding to a double eigenvalue $\omega_1^2(p_0) = \omega_2^2(p_0) = \tau_1^2 \neq 0$, and v_1^0 an eigenvector of the dual problem

$$A_0^Tv_1^0 = \tau_1^2v_1^0 \quad (A_0^T = A^T(p_0)) \tag{4.2}$$

Proof. It follows from condition 1° that

$$(u_1^0,\ v_1^0) = 0 \tag{4.3}$$

Indeed, this is shown by evaluating the scalar product of the equality $A_0w = \tau_1^2w + u_1^0$, which relates the eigenvector u_1^0 to its adjoint w, with the vector v_1^0 and using the equality $(A_0w,\ v_1^0) = (w,\ A_0^Tv_1^0)$.

Now consider the second equation in (3.5). We shall show that it can be solved for u_1^1. Indeed, by (4.3) the Fredholm alternative holds for this equation. There exists a real matrix $G(p_0) = G_0$ such that

$$u_1^1 = \mu_{11}G_0u_1^0 \tag{4.4}$$

G_0 is the inverse of the matrix $A_0 - \tau_1^2 I$ on the subsurface of vectors such that $(u, u_1^0) = 0$. This condition is obeyed by the vector u_1^1 thanks to the normalization condition (3.3).

In view of (4.4), the condition for the solvability of the third equation of (3.5) gives

$$\mu_{11}^2 (G_0 u_1^0, v_1^0) = \pm i\tau_1 (B_0 u_1^0, v_1^0) \quad (\tau_1 \neq 0) \tag{4.5}$$

The fact that the scalar products $(G_0 u_1^0, v_1^0)$ and $(B_0 u_1^0, v_1^0)$ are real implies that μ_{11} (the coefficient of $\varepsilon^{1/2}$ in the expansion of the squared frequency $\omega_{1,\varepsilon}^2$) is real if and only if condition (4.1) holds.

5. *The asymptotic behaviour of the critical curve - ideal perturbation.*

Let $p^1(\varepsilon) = p_0 + \varepsilon^2 p_2^1 + \varepsilon^4 p_4^1 + \ldots$ be some curve of type (2.3) in the εp plane, with arbitrary coefficients p_2^1, p_4^1, on which system (1.4) is stable. It follows from (4.5) and (4.1) that along this curve $\mu_{11} = 0$, and from (4.4) that $u_1^1 = 0$. Let us assume now that $(G_0 u_1^0, v_1^0) \neq 0$. The third equation in (3.5) is solvable, because the vector v_1^0 is orthogonal to its right-hand member. Then

$$u_1^2 = \mu_{12} G_0 u_1^0 \mp i\tau_1 G_0 B_0 u_1^0 \tag{5.1}$$

Proceeding as in the derivation of Eqs.(3.5), we obtain an equation for μ_{12} by collecting the coefficients of ε^2:

$$L_0 u_1^4 = \mu_{12} u_1^2 + \mu_{14} u_1^0 - p_2^1 A_0^1 u_1^0 \mp i\tau_1 B_0 u_1^2 \mp (1/2)\, i\tau_1^{-1} B_0 u_1^0$$
$$A_0^1 = (A(p))'_{p=p_0}$$

If (5.1) is taken into account, the solvability condition for this equation yields a quadratic equation for μ_{12}:

$$\mu_{12}^2 (G_0 u_1^0, v_1^0) \mp i\tau_1 T_1 \mu_{12} - T_2 = 0$$
$$T_1 = (B_0 G_0 u_1^0, v_1^0) + (G_0 B_0 u_1^0, v_1^0), \quad T_2 = \tau_1^2 (B_0 G_0 u_1^0, v_1^0) + p_2^1 (A_0^1 u_1^0, v_1^0)$$

Analysis of the solutions of this equation shows that the inequality $\operatorname{Im}\omega_{1,\varepsilon} > 0$ necessary for system (1.3) to be stable will hold along the curve $p^1(\varepsilon)$ only if $T_1 > 0$ (<0) and $T_2 > 0$ (<0) when $(G_0 u_1^0, v_1^0) > 0$ (<0); but if $(G_0 u_1^0, v_1^0) > 0$ (<0) and at the same time $T_2 < 0$ (>0), then for sufficiently small ε_0 we have $\operatorname{Im}\omega_{1,\varepsilon} < 0$ in the strip $0 < \varepsilon < \varepsilon_0$. Consequently, at $T_2 = 0$ the curve $p^1(\varepsilon)$ separates the strip into a region in which $\operatorname{Im}\omega_{1,\varepsilon} < 0$ and a region in which $\operatorname{Im}\omega_{1,\varepsilon} > 0$. The condition $T_2 = 0$ combined with $(A_0^1 u_1^0, v_1^0) \neq 0$ is equivalent to the equality $p_2^1 = -\tau_1^2 (B_0 G_0 B_0 u_1^0, v_1^0)/(A_0^1 u_1^0, v_1^0)$.

We will now investigate the behaviour along the curve $p^1(\varepsilon)$ of the other (simple) frequencies $\omega_{j,\varepsilon}$ $(j = 3, 4, \ldots, n)$. To this end we use (3.6). The solvability condition for the second equation of (3.6) yields the relation $\mu_{j1}(u_j^0, v_j^0) = \pm i\tau_j (B_0 u_j^0, v_j^0)$ $(j = 3, 4, \ldots, n)$. A necessary condition for the condition $\operatorname{Im}\omega_{j,\varepsilon} > 0$ to hold in a sufficiently small strip $0 < \varepsilon < \varepsilon_0$ is that

$$(B_0 u_j^0, v_j^0)(u_j^0, v_j^0) > 0, \quad j = 3, 4, \ldots, n$$

Summing up, we obtain

Proposition 2. Assume that the necessary condition (4.1) for the perturbation to be ideal is satisfied; assume moreover that $(A_0^1 u_1^0, v_1^0) \neq 0$, $(G_0 u_1^0, v_1^0) > 0$ (<0) and in addition

$$T_1 = (B_0 G_0 u_1^0, v_1^0) + (G_0 B_0 u_1^0, v_1^0) > 0 \quad (<0) \tag{5.2}$$

$$(B_0 u_j^0, v_j^0)(u_j^0, v_j^0) > 0, \qquad j = 3, 4, \ldots, n \tag{5.3}$$

Then for sufficiently small ε_0 the critical curve emanating from the point p_0 and separating the asymptotic stability region of system (1.3) from its unstable region has the following representation in the strip $0 < \varepsilon < \varepsilon_0$:

$$p_\varepsilon = p_0 - \frac{\tau_1^2 (B_0 G_0 B_0 u_1^0, v_1^0)}{(A_0^1 u_1^0, v_1^0)} \varepsilon^2 + o(\varepsilon^2) \tag{5.4}$$

It is worth noting that (5.4) remains formally valid even in the singular case $(G_0 u_1^0, v_1^0) = 0$

6. *The structure of matrices realizing ideal perturbations.*

Let l_1 and l_2 be real numbers. Consider the set of matrices B_0 for which $B_0 u_1^0 = l_1 u_1^0$, $B_0^T v_1^0 = l_2 v_1^0$. Eq.(3.1) is then valid, since condition (4.3) holds. A sufficient condition for the critical curve from a point p_0 to exist in a sufficiently small strip $0 < \varepsilon < \varepsilon_0$ is that inequalities (5.2), (5.4) be valid. We have $T_1 = (l_1 + l_2)(G_0 u_1^0, v_1^0)$. Consequently, inequality (5.2) will hold if the sign of the sum $l_1 + l_2$ is properly chosen. The number of free prameters (elements) of B_0 is $n^2 - 2n$. Choosing them in a suitable manner, one can ensure the validity of the remaining $n - 2$ inequalities (5.3) for $n > 2$.

In the case $n = 2$ conditions 1° and 2° will be satisfied if $k_1 k_2 + 1 = 0$, $k_1 k_2 \neq 0$, $k_1 \neq k_2$, where $k_1 = 2a_{12}/(a_{22} - a_{11})$, $k_2 = 2a_{21}/(a_{22} - a_{11})$, where a_{ij} are the elements of the matrix $A(p_0)$ $(i, j = 1, 2)$. In this case

$$\lambda_1 = \lambda_2 = (a_{11} + a_{22})/2, \quad u_1^0 = (k_1, 1), \quad v_1^0 = (k_2, 1)$$

As $(u_1^0, v_1^0) = 0$, it follows from condition (4.1) that u_1^0, v_1^0 are eigenvectors of B_0, B_0^T, respectively. Hence we arrive at a system of equations for the elements of B_0:

$$b_{11}k_1 + b_{12} = l_1 k_1, \quad b_{11}k_2 + b_{21} = l_2 k_2, \quad b_{12}k_2 + b_{22} = l_2, \quad b_{21}k_1 + b_{22} = l_1$$

The condition $k_1 k_2 + 1 = 0$ for this equation to be solvable is satisfied. The general solution is

$$b_{11} = t, \quad b_{12} = k_1(l_1 - t), \quad b_{21} = k_2(l_2 - t), \quad b_{22} = l_1 + l_2 - t$$

where t, l_1, l_2 are real numbers. For inequality (5.2) to hold we must choose the sign of $l_1 + l_2$ in accordance with the sign of the scalar product $(G_0 u_1^0, v_1^0)$.

To construct G_0, we need only solve the problem $(A_0 - \lambda) u = f$ subject to the condition $(f, v_1^0) = 0$, $(u, u_1^0) = 0$. Defining $u = Cv_1^0$, we see that $C = (f, u_1^0)/(A_0 v_1^0, u_1^0)$. The equality $u = G_0 f$ becomes $u = ((f, u_1^0)/(A_0 v_1^0, u_1^0))\, v_1^0$. Hence it follows that

$$(G_0 u_1^0, v_1^0) = (1 + k_1^2)(1 + k_2^2)(a_{22} - a_{11})/(a_{21} - a_{12})^2$$

Thus, if $a_{22} - a_{11} > 0$ (< 0), the sum $l_1 + l_2$ must be positive (negative).

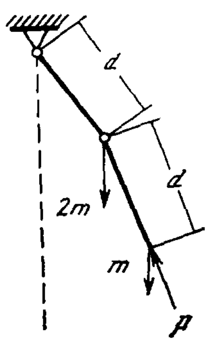
When $t = (k_1 l_2 - k_1 l_1)/(k_2 - k_1)$ we obtain the general structure of symmetric matrices realizing an ideal perturbation. The asymptotic formula (5.4) is then

$$p_\varepsilon = p_0 - \frac{\tau_1^2 l_1 l_2 (G_0 u_1^0, v_1^0)}{(A_0^1 u_1^0, v_1^0)} \varepsilon^2 + o(\varepsilon^2) \tag{6.1}$$

Note that previous publications /10, 11/ described only a few special classes of matrices realizing ideal perturbations.

Example. Let us consider one of the versions of Ziegler's problem of the action of a supporting force on a system of rods with two degrees of freedom /12/ (see the figure). The equation of state can be reduced to the form (1.1), with the matrix $A(p)$ defined as follows:

$$A(p) = \frac{1}{2m^2 d^5} \begin{Vmatrix} 3r - p & p - 2r \\ p - 5r & 4r - p \end{Vmatrix}, \quad r = \frac{c}{d} \tag{6.2}$$



At $p_0 = r(7 - 2\sqrt{2})/2$ the eigenvalue (1.2) is double: $\tau_1^2 = \omega_1^2 = \omega_2^2 = \sqrt{2}\,(\omega_1^2 = 2m^2 d^5 \lambda)$, and corresponding to it we have one eigenvector $u_1^0 = (3 - 2\sqrt{2}, 1)$. The eigenvector of the adjoint problem (4.2) is $v_1^0 = (-(3 + 2\sqrt{2}), 1)$, i.e., $k_2 = -(3 + 2\sqrt{2})$, $k_1 = 3 - 2\sqrt{2}$. To simplify the computations we take $r = 1$. The structure of the ideal symmetric perturbations defined above leads to matrices of the form

$$B_0 = \begin{Vmatrix} b_- & l_1 - l_2 \\ l_1 - l_2 & b_+ \end{Vmatrix}, \quad b_\pm = 3(l_1 \pm l_2) \pm 2\sqrt{2}\,(l_1 - l_2)$$

Since $(A_0^1 u_1^0, v_1^0) = -4\sqrt{2}$ and $(G_0 u_1^0, v_1^0) = 1/4$, the number $l_1 + l_2$ must be positive. The asymptotic critical curve given by (6.1) is

$$p_\varepsilon = (7 - 2\sqrt{2})/2 + l_1 l_2 \varepsilon^2/16 + o(\varepsilon^2)$$

7. *The necessary condition for defective perturbations. Computatation of the defect.*

Proposition 3. Let B be a matrix realizing a defective perturbation of system (1.1). Then necessarily

$$(u_i^0, v_i^0)(B_d u_i^0, v_i^0) \geqslant 0, \quad i = 1, 2, \ldots, n \tag{7.1}$$

where u_i^0 and v_i^0 are solutions of the eigenvalue problem

$$A_d u_i^0 = \omega_i^2 u_i^0, \quad A_d^T v_i^0 = \omega_i^2 v_i^0, \quad A_d = A(p_d), \quad B_d = B(p_d)$$

The proof of Proposition 3 relies on the arguments already invoked to prove inequality (5.3) in Proposition 2, with p_0 replaced by p_d.

We also assume that for all points $p_d < p_0$ such that inequality (7.1) holds, it is true that $(u_i^0, v_i^0) \neq 0$. The case $(u_i^0, v_i^0) = 0$ leads to the previously considered condition $(B_0 u_i^0, v_i^0) = 0$.

Corollary 1. Let $p^1(\varepsilon) = p_d^1 + \varepsilon^2 p_2^1 + \ldots$ be some curve of type (2.3) in the εp plane, emanating from a point p_d^1 on the p-axis. If the strict inequality (7.1) (the inequality inverse to (7.1)) holds at $p = p_d^1$, then for sufficiently small ε_0 system (1.3) is asymptotically stable (unstable) in the strip $0 < \varepsilon < \varepsilon_0$ along the curve $p^1(\varepsilon)$.

Now, considering the matrix $B(p)$ as given, let $Q_B(p)$ denote the set of all p such that

$$Q_B(p) = \{p : p < p_0, (u(p), v(p))(B(p)u(p), v(p)) \geqslant 0\} \tag{7.2}$$

where $u(p)$ and $v(p)$ are eigenvectors of the problems

$$A(p)u(p) = \omega^2(p)u(p),\ A^T(p)v(p) = \omega^2(p)v(p) \tag{7.3}$$
$$(u(p),\ v(p)) \neq 0$$

Let $p^*(B)$ be the supremum of all $p \in Q_B(p)$. If $Q_B(p)$ is empty we define $p^*(B) = -\infty$.

Proposition 4. The defect of the matrix $B(p)$ is given by

$$d_A(B) = p_0 - p^*(B) \quad (p_d = p^*(B))$$

Proof. It follows from the definition of $Q_B(p)$, equalities (7.2) and (7.3) and the previous result that $p_d \in Q_B(p)$. The point p_d cannot be an isolated point of $Q_B(p)$. Indeed, otherwise condition (7.1) would reduce to an equality, while the reverse inequality would hold in a sufficiently small neighbourhood on either side of p_d. But then it would follow from Corollary 1 that the curve (2.3) does not separate the stable from the unstable region, contrary to its definition.

By the definition of $p^*(B)$, we have $p_d \leqslant p^*(B)$. Let $p_d < p^*(B)$ and $\delta = p^*(B) - p_d > 0$. Since p_d is not an isolated point of $Q_B(p)$, there exists a point $p_d^1 \in Q_B(p)$, $p_d < p_d^1 < p^*(B)$, at which inequality (7.1) holds strictly. By Corollary 1, for sufficiently small ε_0 system (1.3) is asymptotically stable in the strip $0 < \varepsilon < \varepsilon_0$ along the curve $p_\varepsilon^1 = p_d^1 + \varepsilon^2 p_2^1 + \ldots$. But since $p_\varepsilon^1 > p_\varepsilon$ (p_ε was defined in (2.3)) in a fairly narrow strip $0 < \varepsilon < \varepsilon_0$, this contradicts the definition of the critical curve p_ε.

Example. Consider the problem of the action of a controlling force on a rod system. Following Ziegler /1/, we assume that the hinges possess given viscoelastic properties. This means that we are considering system (1.1) with the matrix $A(p)$ defined in (6.2) together with terms in $\varepsilon B x^{\cdot}(t)$, where $B = \|b_{ij}\|$, $b_{11} = 3/2$, $b_{12} = -1$, $b_{21} = -5/2$, $b_{22} = 2$. It was shown in Sect.6 that $p_0 = 2.0857r$ $(r = c/d)$. It can be verified directly that condition (4.1) fails to hold, and therefore the perturbation realized by B is defective.

Computations using the Routh-Hurwitz criterion give a value of $p_d = 1.4642$. Let us calculate this number using the result of Proposition 4. To simplify matters, we put $r = 1$. The eigenvectors of problems (7.3) are

$$u_i = (1, z_i/(2-p)),\quad v_i = (1, z_i/(5-p))$$
$$z_i = 2\omega_i^2(p) - p + 3,\quad i = 1, 2 \quad (p \neq 2,\ p \neq 5)$$

Note that $(u_i(p), v_i(p)) > 0$ for $p < 2$. With these values of p, a sufficient condition for inequalities (7.2) to hold is the validity of at least one of the two pairs of inequalities $24 - 7p \geqslant \sqrt{D_2} \mp 4\sqrt{D_1}$, $24 - 7p \leqslant \mp(\sqrt{D_2} + 4\sqrt{D_1})$, where $D_1 = 4p^2 - 28p + 41$, $D_2 = p^2 + 56p - 80$. The domain of definition of these inequalities (accurate to four decimal places) is $p \leqslant -37.3938$, $1.3938 \leqslant p \leqslant 2.0857$. The second of the first pair of inequalities (with the plus sign chosen) does not hold for any p in the domain of definition. The supremum of the solutions of the remaining inequalities is 1.4642. In the region $2 < p < 2.0857$ we have $(u_i(p), v_i(p)) < 0$ $(i = 1, 2)$ and inequalities (7.2) do not hold for such values of p. In sum, $p_d = p^*(B) = 1.4642$, agreeing with the previously known value.

Note that in the case $n = 2$ one can derive general formulae, based on the elements of the matrices A and B. Indeed, repeating the arguments of Sect.2 we obtain

$$R(\varepsilon^2, p) = (a_{11}a_{22} - a_{12}a_{21})(b_{11} + b_{22})^2 - (a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12})[(a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}) + (b_{11} + b_{22})(b_{11}b_{22} - b_{12}b_{21})\varepsilon^2] \tag{7.4}$$

$$\operatorname{Im}\left(\frac{\partial\omega_j}{\partial\varepsilon}\right)_{\varepsilon=0} = \frac{1}{4}\{(b_{11} + b_{22}) \pm \delta\Delta^{-1/2}\} \quad (p < p_0) \tag{7.5}$$

$$\delta = 2(a_{12}b_{21} + a_{21}b_{12}) + (a_{11} - a_{22})(b_{11} - b_{22})$$
$$\Delta = (a_{11} - a_{22})^2 + 4a_{12}a_{21}$$
$$(\omega_{1,2})_{\varepsilon=0} = \pm\sqrt{\lambda_1(p)},\quad (\omega_{3,4})_{\varepsilon=0} = \pm\sqrt{\lambda_2(p)},\quad 0 < \lambda_1(p) < \lambda_2(p)$$

λ_1 and λ_2 are the eigenvalues of the unperturbed problem (1.2) at $p_d \leqslant p_0$. The upper sign in formula (7.5) pertains to λ_2, and the lower one to λ_1. By (7.5), if

$$|\delta| < (b_{11} + b_{22})\,\Delta^{1/2} \tag{7.6}$$

for $p < p_0$, then for all $(\varepsilon, p)\ (\varepsilon > 0)$ sufficiently close to $(0, p)$ system (1.3) is asymptotically stable, but if (7.6) is reversed, then the system is unstable.

Let us assume that the set of solutions of the equations $R(\varepsilon^2, p) = 0$ in some strip $0 \leqslant \varepsilon \leqslant \varepsilon_0$ is bounded. Then, if inequality (7.6) is true for all $p < p_0$, the perturbation in question is ideal; but if the reverse inequality holds for some $p < p_0$ then p_d is the supremum of those p values for which (7.6) is true (if there are no such values, then $p_d = -\infty$). Note that condition 1° of Sect.1 implies the following relations

$$\begin{aligned} &\Delta > 0,\ a_{11} + a_{22} > 0,\ a_{11}a_{22} - a_{12}a_{21} > 0 \quad (p < p_0)\\ &\Delta = 0,\ a_{12} \neq a_{21} \qquad (p = p_0)\\ &\Delta < 0 \qquad (p > p_0)\end{aligned} \tag{7.7}$$

Since the right-hand side of (7.6) vanishes at $p = p_0$, it follows that if

$$\delta|_{p=p_0} \neq 0 \tag{7.8}$$

the perturbation is defective. Thus, defective perturbations are actually the typical situation when condition 1° is satisfied. This can also be derived from condition (4.1).

It is not hard to show that for any matrix A satisfying conditions (7.7) one can find a symmetric positive-definite matrix B such that inequality (7.8) is true. In other words, mechanical systems of the type considered here may be destabilized by arbitrarily small dissipative forces with total dissipation. It should also be noted that along any critical curve, necessarily.

$$\operatorname{sgn}(a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12}) = \operatorname{sgn}(b_{11} + b_{22})$$

Thus, in the above example, condition (7.6) becomes $|14p - 41| < 7(4p^2 - 28p + 41)^{1/2}$. This inequality is true only provided that $p < 41/28$; thus $p_d = 41/28 = 1.4643$. The same follows from (7.4): $R(\varepsilon^2, p) = 0$ when $p_d = 41/28$.

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